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The mean-field scaling function of the universality class of absorbing phase transitions with a conserved field

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Abstract
We consider two mean-field like models which belong to the universality class of absorbing phase transitions with a conserved field. In both cases we analytically derive the order parameter as a function of the control parameter and of an external field conjugated to the order parameter. This allows us to calculate the universal scaling function of the mean-field behaviour. The obtained universal function is in perfect agreement with recently obtained numerical data of the corresponding five- and six-dimensional models, showing that four is the upper critical dimension of this particular universality class.

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1. Introduction

The scaling behaviour of directed percolation is recognized as the paradigmatic example of the critical behaviour of several non-equilibrium systems which exhibits a continuous phase transition from an active state to an absorbing non-active state (see, for instance, [1, 2]). The widespread occurrence of such systems in physics, biology, as well as catalytic chemical reactions, is reflected by the well known universality hypothesis of Janssen and Grassberger that models exhibiting continuous phase transition to a single absorbing state generally belong to the universality class of directed percolation [3, 4]. Introducing an additional symmetry the critical behaviour differs from directed percolation. In particular, particle conservation leads to a new universality class of absorbing phase transitions with a conserved field as pointed out in [5]. In this work the authors introduce two models, the conserved lattice gas (CLG) model, as well as a conserved threshold transfer process (CTTP). The latter is a conserved modification of the threshold transfer process introduced in [6]. Both models display a continuous phase transition from an active to an inactive phase. The density of active sites ρ_a is the order parameter of the phase transition controlled by the total density of particles ρ, i.e. $\rho_a > 0$ if the density exceeds the critical value ρ_c and zero otherwise. As usual in second-order phase transitions the order parameter vanishes algebraically at the transition point. The corresponding order parameter exponent as well as the exponent of the order parameter fluctuations of the CLG are determined in [7] for various dimensions.

The scaling behaviour of the CLG model in an external field conjugated to the order parameter was considered recently [8]. The external field is realized by movements of inactive particles which may be activated in this way. Thus the field creates active particles without violating particle conservation. Taking into account this additional scaling field the order parameter obeys the scaling ansatz

$$\rho_a(\delta\rho, h) \sim \lambda \tilde{r}(\delta\rho\lambda^{-1/\beta}, h\lambda^{-\sigma/\beta}) \tag{1}$$

with the critical exponents β and σ, the scaling function $\tilde{r}$, the reduced control parameter $\delta\rho = \rho/\rho_c - 1$ and the external field h. Choosing $\delta\rho\lambda^{-1/\beta} = 1$, for zero fields one gets $\rho_a \sim \tilde{r}(1,0)\delta\rho^{\beta}$, whereas $h\lambda^{-\sigma/\beta} = 1$ leads at the critical density to $\rho_a \sim \tilde{r}(0,1)h^{\beta/\sigma}$. Except for the critical point ($\delta\rho = 0, h = 0$), the scaling function $\tilde{r}(x, y)$ is smooth and analytic but it is not universal since it may depend, like the value of ρ_c, on the details of the considered systems (here, for example, the lattice structure, the update scheme, etc).

A universal scaling function $\tilde{R}$ can be introduced if one allows non-universal metric factors c_i for the scaling arguments $\delta\rho$ and h (see, for instance, [9]), i.e.

$$\rho_a(\delta\rho, h) \sim \lambda \tilde{R}(c_1\delta\rho\lambda^{-1/\beta}, c_2 h\lambda^{-\sigma/\beta}), \tag{2}$$

and the scaling function is normed by the conditions $\tilde{R}(1,0) = \tilde{R}(0,1) = 1$. Then the function $\tilde{R}(x, y)$ is universal, i.e. similar to the critical exponents, and $\tilde{R}(x, y)$ is identical for all models which belong to the same universality class. But the non-universal metric factors differ again between the models and may depend on the lattice structure, the used update scheme, etc.

The non-universal metric factors can be easily determined by the scaling behaviour of the order parameter at zero field and at the critical density, respectively. Choosing $c_1\delta\rho\lambda^{-1/\beta} = 1$ for zero fields ($h = 0$) one gets

$$\rho_a(\delta\rho, 0) \sim (c_1\delta\rho)^{\beta} \tag{3}$$

whereas $c_2 h\lambda^{-\sigma/\beta} = 1$ leads at the critical density ($\delta\rho = 0$) to

$$\rho_a(0, h) \sim (c_2 h)^{\beta/\sigma}. \tag{4}$$

In this work we derive the universal scaling function $\tilde{R}$ of the mean-field solution of the universality class of absorbing phase transitions with a conserved field. In particular, we consider analytically the CLG and the CTTP with particle hopping to randomly chosen sites on the whole lattice. This unrestricted particle hopping breaks long-range correlations and the scaling behaviour is characterized by the mean-field exponents (see [10]). Neglecting correlations it is possible to analytically derive the order parameter as a function of the control parameter and of the external field. The obtained universal function is in perfect agreement with recently obtained numerical data of the five- and six-dimensional CLG and CTTP in an external field.

2. The conserved lattice gas

We consider the CLG model on a chain with L sites and periodic boundary conditions. At the beginning one randomly distributes $N = \rho L$ particles on the system, where ρ denotes the particle density. A particle is called active if at least one of its two neighbouring sites is occupied. In the original CLG model active particles jump in the next update step to one of

Table 1. The configuration of a CLG lattice before ($\mathcal{C}$) and after ($\mathcal{C}'$) a particle hopping. Only the target lattice site where a particle hops onto and its left- and right-neighbouring sites are shown. Empty sites are marked by $\circ$, inactive sites by $*$ and active sites by $\bullet$. Δn denotes the change of the number of active sites due to the particle hopping and p is the corresponding probability of the configuration $\mathcal{C}$ if one neglects spatial correlations.

$\mathcal{C}$	$\mathcal{C}'$	Δn	$p(\mathcal{C} \to \mathcal{C}')$
$\circ\circ\circ$	$\circ * \circ$	-1	$\rho_a(1-\rho)(1-\rho)^2$
$*\circ\circ$	$\bullet\bullet\circ$	+1	$\rho_a(1-\rho)2\rho_i(1-\rho)$
$*\circ*$	$\bullet\bullet\bullet$	+2	$\rho_a(1-\rho)\rho_i^2$
$\bullet\circ\circ$	$\bullet\bullet\circ$	0	$\rho_a(1-\rho)2\rho_a(1-\rho)$
$\bullet\circ\bullet$	$\bullet\bullet\bullet$	0	$\rho_a(1-\rho)\rho_a^2$
$\bullet\circ*$	$\bullet\bullet\bullet$	+1	$\rho_a(1-\rho)2\rho_a\rho_i$

their empty nearest-neighbour sites, selected at random [5]. In the steady state the system is characterized by the density of active sites ρ_a which depends on ρ. The density of inactive sites is given by $\rho_i = \rho - \rho_a$ and $1 - \rho$ is the density of empty sites.

We introduced in [10] a modification of the CLG model where active particles are moved to a randomly chosen empty lattice site which suppresses long-range correlations. A given lattice site is active with a probability ρ_a and with the probability $1 - \rho$ it may be moved to an empty lattice site. Depending on the neighbourhood of this new lattice site the number of active sites may change. For instance, if both new neighbours of the moved particle are empty the number of active particles is reduced by one ($\Delta n = -1$). Without correlations the corresponding probability for this process is $\rho_a(1-\rho)^3$. In the case that one of the new neighbours of the moved particle is occupied by an inactive particle (ρ_i) and the second neighbour is empty ($1-\rho$), the number of active sites is increased by one ($\Delta n = 1$). The corresponding probability is given by $p = 2\rho_a\rho_i(1-\rho)^2$. All other possible configurations and the corresponding probabilities are listed in table 1.

The probabilities that the number of active particles are changed by Δn are given by

$$\begin{aligned} p_{\Delta n=-1} &= (1-\rho)\rho_a(1-\rho)^2, \\ p_{\Delta n=0} &= (1-\rho)\rho_a[2\rho_a(1-\rho)+\rho_a^2], \\ p_{\Delta n=1} &= (1-\rho)\rho_a[2\rho_i(1-\rho)+2\rho_a\rho_i], \\ p_{\Delta n=2} &= (1-\rho)\rho_a\rho_i^2. \end{aligned} \tag{5}$$

The expectation value of Δn is

$$E[\Delta n] = \sum_{\Delta n=-1}^{2} \Delta n p_{\Delta n} = (1-\rho)\rho_a[-1-2\rho_a+4\rho-\rho^2]. \tag{6}$$

As pointed out in [10], the average number of active sites is constant in the stationary state, i.e. the expectation value of Δn should be zero in the steady state. Using the constraint $E[\Delta n] = 0$ one gets

$$\rho = 1 \vee \rho_a = 0 \vee -1 - 2\rho_a + 4\rho - \rho^2 = 0. \tag{7}$$

The first equation corresponds to a system were all sites are occupied ($\rho_a = 1$) and no dynamics can take place, whereas the absorbing state is reflected by the second equation. The non-trivial third equation corresponds, for $\rho_a > 0$, to the active phase and one gets for the order parameter in leading order

$$\rho_a = \frac{4\rho - \rho^2 - 1}{2} = (2\sqrt{3} - 3)\delta\rho + O(\delta\rho^2) \tag{8}$$

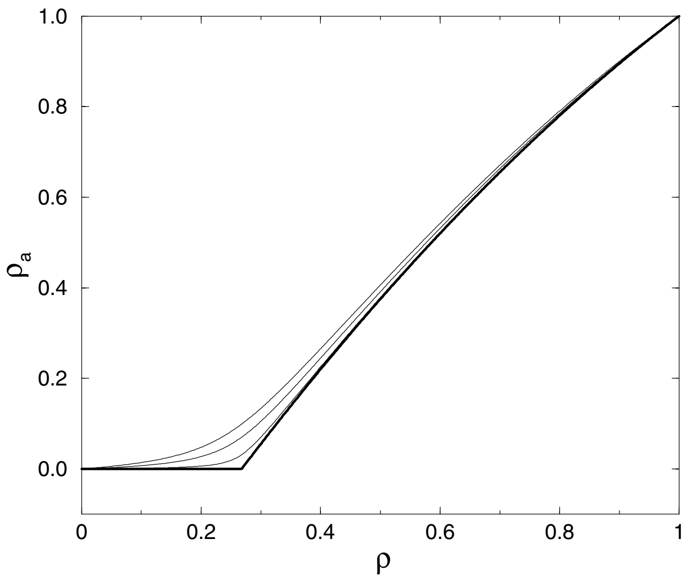

Figure 1. The order parameter of the CLG model as a function of the particle density ρ and the applied external field h (see equation (11)). The thick curve corresponds to $h = 0$ whereas the thin curves correspond to $h = 0.1, 0.05, 0.01$ (from top to bottom).

with the critical density $\rho_c = 2 - \sqrt{3}$ [10]. Thus we have obtained the critical exponent $\beta = 1$ as well as the non-universal metric factor $c_1 = 2\sqrt{3} - 3$.

In the case that an external field is applied non-active sites may be activated (see [8]). The probability that a site is occupied and has two empty neighbours is $\rho(1 - \rho)^2$. These particles are activated with probability h, where h denotes the strength of the applied field. In this process the number of active sites is increased ($\Delta n = 1$) and the probability $p_{\Delta n=1}$ is modified to

$$p_{\Delta n=1} = (1 - \rho)\rho_a[2(1 - \rho)\rho_i + 2\rho_i\rho_a] + (1 - \rho)^2\rho h. \tag{9}$$

Using again the steady state condition $E[\Delta n] = 0$ one gets the equations

$$\rho = 1 \vee \rho_a[-1 - 2\rho_a + 4\rho - \rho^2] + (1 - \rho)\rho h = 0. \tag{10}$$

The first equation again corresponds to the trivial case of a totally occupied lattice whereas the second equation yields the solutions

$$\rho_a = \tfrac{1}{4}\Big(-1 + 4\rho - \rho^2 \pm \sqrt{8h(1 - \rho)\rho + (-1 + 4\rho - \rho^2)^2}\Big). \tag{11}$$

The solution with the + sign describes the order parameter $\rho_a(\rho, h)$ as a function of the density and of the external field whereas the $-$ sign solution yields negative densities for the order parameter for all values of ρ and h. A sketch of the order parameter for various fields is presented in figure 1.

At the critical density $\rho_c = 2 - \sqrt{3}$ the order parameter is given by

$$\rho_a(\rho_c, h) = \sqrt{\frac{3\sqrt{3} - 5}{2}}\sqrt{h} \tag{12}$$

i.e. the field scaling exponent is $\sigma = 2$ and the non-universal metric factor is $c_2 = (3\sqrt{3}-5)/2$.

In the following we derive the universal scaling function $R(x, y)$ of the mean-field solution. Therefore, we write the order parameter (equation (11)) as a function of the reduced control parameter $\delta\rho$ and consider the function $\rho_a(\delta\rho, h)/\sqrt{h}$. Since we are interested in the scaling behaviour in the vicinity of the critical point we perform the limits $\rho_a \to 0$, $\delta\rho \to 0$ and $h \to 0$, with the constraint that $\rho_a/\sqrt{h}$ and $\delta\rho/\sqrt{h}$ are finite. Thus all terms which scale as $\delta\rho^2/\sqrt{h}$ or $\delta\rho\sqrt{h}$ vanish in leading order and we get

$$\frac{\rho_a(\delta\rho, h)}{\sqrt{h}} = \frac{2\sqrt{3}-3}{2}\frac{\delta\rho}{\sqrt{h}} + \sqrt{\frac{3\sqrt{3}-5}{2} + \left(\frac{2\sqrt{3}-3}{2}\frac{\delta\rho}{\sqrt{h}}\right)^2}. \tag{13}$$

Introducing the non-universal metric factors $c_1 = 2\sqrt{3} - 3$ and $c_2 = (3\sqrt{3} - 5)/2$ one gets the universal function

$$\tilde{R}(c_1\delta\rho, c_2 h) = \frac{c_1\delta\rho}{2} + \sqrt{c_2 h + \left(\frac{c_1\delta\rho}{2}\right)^2}. \tag{14}$$

Equations (8), (12) are recovered from this result by setting $h = 0$ and $\delta\rho = 0$, respectively. Furthermore, we get $\tilde{R}(1, 0) = \tilde{R}(0, 1) = 1$ as required above.

As usual in scaling analysis (see, for instance, [8]) the order parameter as well as the control parameter are rescaled by the field in order to obtain a data collapse (setting $c_2 h\lambda^{-\sigma/\beta} = 1$ in equation (2)). In this case one gets the universal function

$$\frac{\rho_a(\delta\rho, h)}{\sqrt{c_2 h}} \sim \tilde{R}(x, 1) = \frac{x}{2} + \sqrt{1 + \left(\frac{x}{2}\right)^2} \tag{15}$$

where the scaling argument is given by $x = c_1\delta\rho/\sqrt{c_2 h}$.

For the sake of simplicity we derived the scaling function of the one-dimensional CLG model only. A straightforward extension to higher-dimensional systems for $h = 0$ has already been presented in [10]. The increased number of nearest neighbours in higher dimensions affects the non-universal quantities ρ_c, c_1 and c_2 only, but not the critical exponents and the universal scaling function.

3. The conserved threshold transfer process

A similar analysis can be performed for the CTTP with random neighbour hopping. In the CTTP lattice sites may be empty, occupied or doubly occupied. Doubly occupied lattice sites are considered as active and one tries to transfer both particles of each active site to randomly chosen lattice sites. Recently performed numerical investigations in dimensions $d = 2, 3, 4, 5, 6$ confirm the conjecture of [5] that the CLG and the CTTP belong to the same universality class [11]. Analogous to the above-presented analysis we derive the mean-field critical behaviour of the order parameter of the CTTP with random neighbour hopping.

In the following we denote the densities of sites with ρ_a (doubly occupied and active), ρ_i (singly occupied and inactive) and ρ_e (empty). Normalization requires $\rho_e + \rho_i + \rho_a = 1$ and the particle conservation is reflected by the equation $\rho_i + 2\rho_a = \rho$, where the control parameter ρ again denotes the density of particles on a D-dimensional lattice, i.e. $\rho = N/L^D$. The probability that a given lattice site s is active is therefore ρ_a. In this case the two active particles are tried to transfer to two randomly chosen lattice sites t_1 and t_2. In the case that both sites are empty the two particles are moved to the empty sites and the number of active sites is decreased by one ($\Delta n = -1$). The probability for this process is $\rho_a\rho_e^2$. All other possible

Table 2. The configuration of a CTTP lattice before (s, t_1, t_2) and after (s', t_1', t_2') a particle hopping. Only the source lattice site (s) and its two targets sites (t_1 and t_2) where the two particles may be moved are shown. Δn denotes the change of the number of active sites due to the particle hopping and p is the corresponding probability of the configuration (s, t_1, t_2) if one neglects spatial correlations.

s	t_1	t_2	s'	t_1'	t_2'	Δn	$p(s, t_1, t_2)$
2	0	0	0	1	1	−1	$\rho_a\rho_e^2$
2	0	1	0	1	2	0	$\rho_a 2\rho_e\rho_i$
2	0	2	1	1	2	−1	$\rho_a 2\rho_e\rho_a$
2	1	1	0	2	2	+1	$\rho_a\rho_i^2$
2	1	2	1	2	2	0	$\rho_a 2\rho_i\rho_a$
2	2	2	2	2	2	0	$\rho_a\rho_a^2$

configurations and the corresponding probabilities are listed in table 2. The probabilities that the number of active particles are changed by Δn are thus given by

$$
\begin{aligned}
p_{\Delta n=-1} &= \rho_a[\rho_e^2 + 2\rho_e\rho_a] \\
p_{\Delta n=0} &= \rho_a[2\rho_e\rho_i + 2\rho_i\rho_a + \rho_a^2] \\
p_{\Delta n=1} &= \rho_a\rho_i^2.
\end{aligned}
\tag{16}
$$

The steady state condition $E[\Delta n] = 0$ leads to the equations

$$\rho_a = 0 \vee -1 + 2\rho - 4\rho_a + \rho_a^2 = 0. \tag{17}$$

Again the first equation corresponds to the absorbing state and the second equation yields the order parameter as a function of the particle density

$$\rho_a = 2 \pm \sqrt{5 - 2\rho}. \tag{18}$$

Here, the + solution can be neglected ($\rho_a > 1$) and the − solution describes the order parameter behaviour above the critical density $\rho_c = 1/2$. Close to this critical point the order parameter scales in leading order as

$$\rho_a = \tfrac{1}{4}\delta\rho + O(\delta\rho^2), \tag{19}$$

i.e. the non-universal metric factor of the CTTP is $c_1 = 1/4$ and the critical exponent is in agreement with the CLG model $\beta = 1$.

Similar to the CLG model we now apply an external field which activates singly occupied sites. The probability that the external field h acts to a given site is $\rho_i h$ and one tries to transfer this particle to a randomly chosen lattice site. In the case that the activated particle is moved to an empty lattice site the number of active site is unchanged by this field-induced process ($\Delta n = 0$). The number of active sites is increased only if the particle is moved to a singly occupied lattice site ($\Delta n = +1$). The probability for this process is $\rho_i^2 h$. In order to incorporate the external field into the dynamics one has to modify $p_{\Delta n=1}$ accordingly, and the steady state condition yields

$$\rho_a(-1 + 2\rho - 4\rho_a + \rho_a^2) + h(\rho - 2\rho_a)^2 = 0. \tag{20}$$

At the critical density $\rho_c = 1/2$ the order parameter scales with the external field according to

$$\rho_a = \tfrac{1}{4}h^{1/2} + O(h), \tag{21}$$

i.e. the critical exponent is again $\sigma = 2$, and the non-universal metric factor of the CTTP is given by $c_2 = 1/16$.

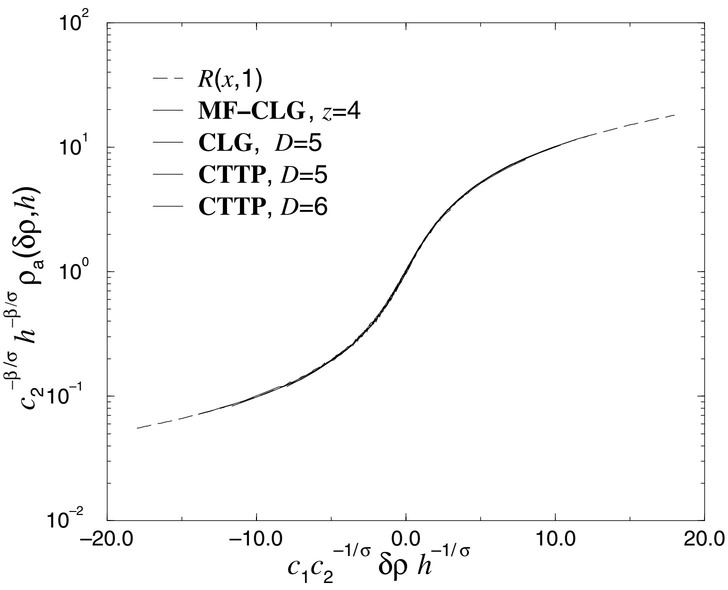

Figure 2. The mean-field universal function $\tilde{R}(x, 1)$ (see equation (15)) of the universality of absorbing phase transitions with a conserved field. The numerical data of the five- and six-dimensional models are obtained from [8, 11]. Additionally, we plot the data of a (mean-field like) CLG model with random neighbour hopping on a square lattice ($z = 4$ next neighbours) which was introduced in [7]. At least four different field values are plotted for each model.

In order to obtain the universal scaling function $\tilde{R}$, we set $\rho = \rho_c + \rho_c\delta\rho$ and transform equation (20) into

$$\frac{\rho_a}{\sqrt{h}}\left(\frac{\delta\rho}{\sqrt{h}} - 4\frac{\rho_a}{\sqrt{h}} + \frac{\rho_a^2}{\sqrt{h}}\right) + \left(\frac{1}{2} + \frac{1}{2}\delta\rho - 2\rho_a^2\right)^2 = 0. \tag{22}$$

Focusing on the critical scaling behaviour ($h \to 0$, $\delta\rho \to 0$, $\rho_a \to 0$ where again $\rho_a/\sqrt{h}$ as well as $\delta\rho/\sqrt{h}$ is kept constant) we can neglect all irrelevant terms and get in leading order

$$\frac{\rho_a}{\sqrt{h}}\left(\frac{\delta\rho}{\sqrt{h}} - 4\frac{\rho_a}{\sqrt{h}}\right) + \frac{1}{4} = 0. \tag{23}$$

This equation can be easily solved and one gets

$$\rho_a(\delta\rho, h) = \frac{\delta\rho}{8} \pm \sqrt{\frac{h}{16} + \left(\frac{\delta\rho}{8}\right)^2} \tag{24}$$

where the $-$ sign can be neglected since it yields negative values of the order parameter. Using the non-universal metric factors $c_1 = 1/4$ and $c_2 = 1/16$ once again we eventually get equation (14), i.e. both models the CLG as well as the CTTP are characterized by the same universal function $\tilde{R}(x, y)$ in the mean-field solution. Furthermore, the obtained universal function $\tilde{R}$ agrees with that of the mean-field solution of directed percolation (see, for instance, [12]), i.e. although the CLG and CTTP differ from the directed percolation scaling behaviour in low dimensions they coincide on the mean-field level.

4. Numerical simulations

In the following we compare our results with those obtained from numerical simulations. The upper critical dimension of the universality class of absorbing phase transitions with a conserved field is $D_c = 4$ [7]. Thus we compare our results with the scaling behaviour of the CLG in $D = 5$ [8], the CTTP in $D = 5$ and 6 [11], as well as with the scaling behaviour of a two-dimensional CLG on a square lattice where active particles are moved to randomly chosen lattice sites [10]. In all models the order parameter is determined as a function of the control parameter for various fields and the data are rescaled according to equation (15). Varying the non-universal metric factors we observe a data-collapse with the universal function $\tilde{R}(x, 1)$. The corresponding curves are presented in figure 2. As one can see, all numerically obtained curves fit well with the derived universal function. Furthermore, the perfect data collapse of the curves for different dimensions, as well as for a mean-field model clearly confirms that four is the upper critical dimension.

Notice that the mean-field behaviour of the CTTP order parameter was recently considered in [13]. Using a cluster approximation method the authors obtained equation (18) which describes the zero-field behaviour of the order parameter.

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